

# On Weil restriction of reductive groups and a theorem of Prasad

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**Abstract.** Let  $G$  be a connected simple semisimple algebraic group over a local field  $F$  of arbitrary characteristic. In a previous article by the author the Zariski dense compact subgroups of  $G(F)$  were classified. In the present paper this information is used to give another proof of a theorem of Prasad [8] (also proved by Margulis [3]) which asserts that, if  $G$  is isotropic, every non-discrete closed subgroup of finite covolume contains the image of  $\tilde{G}(F)$ , where  $\tilde{G}$  denotes the universal covering of  $G$ . This result played a central role in Prasad's proof of strong approximation. The present proof relies on some basic properties of Weil restrictions over possibly inseparable field extensions, which are also proved here.

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## 1. Weil restriction of linear algebraic groups

Let  $F$  be a field and  $F'$  a subfield such that  $[F/F'] < \infty$ . In this section we discuss some properties of the Weil restriction  $\mathcal{R}_{F/F'}G$  where  $G$  is a linear algebraic group over  $F$ . We are interested particularly in the case that  $F/F'$  is inseparable, where the Weil restriction involves some infinitesimal aspects. Thus the natural setting is that of group schemes. We assume that  $G$  is a connected affine group scheme of finite type that is smooth over  $F$ . The smoothness condition is equivalent to saying that  $G$  is reduced and “defined over  $F$ ” in the terminology of [11] Ch.11.

Throughout, we will speak of a scheme over a ring  $R$  when we really mean a scheme over  $\mathbf{Spec} R$ . Similarly, for any ring homomorphism  $R' \rightarrow R$  and any scheme  $X'$  over  $R'$  we will abbreviate  $X' \times_{R'} R := X' \times_{\mathbf{Spec} R'} \mathbf{Spec} R$ . The basic facts on Weil restrictions that we need are summarized in [4] Appendix 2–3. Throughout the following we abbreviate

$$G' := \mathcal{R}_{F/F'}G.$$

By [4] A.3.2, A.3.7 this is a connected smooth affine group scheme over  $F'$ . The universal property of the Weil restriction identifies  $G'(F')$  with  $G(F)$ .

Next, we fix an algebraic closure  $E'$  of  $F'$  and abbreviate  $E := F \otimes_{F'} E'$ . With  $\Sigma := \text{Hom}_{F'}(F, E')$  there is then a unique decomposition  $E = \bigoplus_{\sigma \in \Sigma} E_\sigma$ , where each  $E_\sigma$  is a local ring with residue field  $E'$  and the composite map  $F \rightarrow E_\sigma \twoheadrightarrow E'$  is equal to  $\sigma$ . The Weil restriction from any finite dimensional commutative  $E'$ -algebra down to  $E'$  is defined, and by [4] A.2.7–8 we have natural isomorphisms

$$\begin{aligned} G' \times_{F'} E' &\cong \mathcal{R}_{E/E'}(G \times_F E) \\ &= \mathcal{R}_{E/E'}\left(\bigsqcup_{\sigma \in \Sigma} G \times_F E_\sigma\right) \\ &\cong \prod_{\sigma \in \Sigma} G_\sigma \end{aligned} \quad (1.1)$$

with

$$G_\sigma := \mathcal{R}_{E_\sigma/E'}(G \times_F E_\sigma).$$

These isomorphisms are functorial in  $G$  and equivariant under  $\text{Aut}(E'/F')$ , which acts on the right hand side by permuting the factors according to its action on  $\Sigma$ . Next, for every  $\sigma \in \Sigma$  we fix a filtration of  $E_\sigma$  by ideals

$$E_\sigma \supsetneq I_{\sigma,1} \supsetneq \cdots \supsetneq I_{\sigma,q-1} \supsetneq I_{\sigma,q} = 0$$

with subquotients of length 1. Here  $q$  is the degree of the inseparable part of  $F/F'$ . We also choose a basis of every successive subquotient. For every  $1 \leq i \leq q$  there is a natural homomorphism

$$G_\sigma = \mathcal{R}_{E_\sigma/E'}(G \times_F E_\sigma) \longrightarrow \mathcal{R}_{(E_\sigma/I_{\sigma,i})/E'}(G \times_F (E_\sigma/I_{\sigma,i})).$$

Let  $G_{\sigma,i}$  denote its kernel. By [4] A.3.5 we find that each  $G_{\sigma,i}$  is smooth over  $F'$  and there are canonical isomorphisms

$$G_\sigma/G_{\sigma,1} \cong G \times_{F,\sigma} E' \quad (1.2)$$

and

$$G_{\sigma,i}/G_{\sigma,i+1} \cong \text{Lie } G \otimes_{F,\sigma} \mathbb{G}_{a,E'} \quad (1.3)$$

for all  $1 \leq i \leq q-1$ , where  $\mathbb{G}_a$  denotes the additive group of dimension 1. Moreover, this description is functorial in  $G$ . Namely, let  $H$  be another smooth group scheme over  $F$  and define  $H' := \mathcal{R}_{F/F'} H$ ,  $H_\sigma$  and  $H_{\sigma,i}$  in the obvious way. Then any homomorphism  $\varphi: H \rightarrow G$  induces homomorphisms  $\mathcal{R}_{F/F'} \varphi: H' \rightarrow G'$ ,  $H_\sigma \rightarrow G_\sigma$  and  $H_{\sigma,i} \rightarrow G_{\sigma,i}$  and the resulting homomorphisms on subquotients are just

$$\varphi \times \text{id}: H \times_{F,\sigma} E' \longrightarrow G \times_{F,\sigma} E' \quad (1.4)$$

and

$$d\varphi \otimes \text{id}: \text{Lie } H \otimes_{F,\sigma} \mathbb{G}_{a,E'} \longrightarrow \text{Lie } G \otimes_{F,\sigma} \mathbb{G}_{a,E'}. \quad (1.5)$$

Recall that an *isogeny* of algebraic groups is a surjective homomorphism with finite kernel. An isogeny  $\varphi$  is *separable* if and only if its derivative  $d\varphi$  is an isomorphism.

**Proposition 1.6.** *Let  $\varphi: H \rightarrow G$  be a homomorphism of connected smooth linear algebraic groups over  $F$ .*

- (a) *If  $F/F'$  is separable, then  $\mathcal{R}_{F/F'}\varphi: H' \rightarrow G'$  is an isogeny if and only if  $\varphi$  is an isogeny.*
- (b) *If  $F/F'$  is inseparable, then  $\mathcal{R}_{F/F'}\varphi: H' \rightarrow G'$  is an isogeny if and only if  $\varphi$  is a separable isogeny.*

*Proof.* In the separable case we have  $E' \xrightarrow{\sim} E_\sigma$ , and assertion (a) follows directly from the decomposition 1.1 and the functoriality 1.4. So assume that  $F/F'$  is inseparable, i.e., that  $q > 1$ . First note that  $\dim H' = [F/F'] \cdot \dim H$  and  $\dim G' = [F/F'] \cdot \dim G$ , by the successive extension above or by [4] A.3.3. Thus if either  $\varphi$  or  $\mathcal{R}_{F/F'}\varphi$  is an isogeny, we must have  $\dim H = \dim G$ .

If  $\mathcal{R}_{F/F'}\varphi$  is an isogeny, its kernel is finite; hence so is the kernel of its restriction  $H_{\sigma,q-1} \rightarrow G_{\sigma,q-1}$ . By 1.5 this means that  $d\varphi$  is injective. For dimension reasons it follows that  $d\varphi$  is an isomorphism; hence  $\varphi$  is a separable isogeny, as desired.

Conversely, suppose that  $\varphi$  is a separable isogeny. Then all the homomorphisms on subquotients 1.4 and 1.5 induced by  $\mathcal{R}_{F/F'}\varphi$  are surjective. Using the snake lemma inductively one deduces that  $\mathcal{R}_{F/F'}\varphi$  itself is surjective. For dimension reasons it is therefore an isogeny, as desired.  $\square$

**Theorem 1.7.** *If  $G$  is reductive and  $F'$  infinite, then  $G'(F')$  is Zariski dense in  $G'$ .*

*Proof.* If  $F/F'$  is separable, the isomorphism 1.1 shows that  $G'$  is reductive. In that case the assertion is well-known: see [11] Cor.13.3.12 (i).

We will adapt the argument to the general case.

Assume first that  $G = T$  is a torus. Choose a finite separable extension  $F_1/F$  which splits  $T$ , and fix an isomorphism  $\mathbb{G}_{m,F_1}^n \xrightarrow{\sim} T \times_F F_1$ , where  $\mathbb{G}_m$  denotes the multiplicative group of dimension 1. Combining this with the norm map yields a surjective homomorphism

$$\mathcal{R}_{F_1/F}\mathbb{G}_{m,F_1}^n \longrightarrow \mathcal{R}_{F_1/F}(T \times_F F_1) \xrightarrow{\text{Nm}} T.$$

Since  $F_1/F$  is separable, this morphism is smooth. By [4] A.2.4, A.2.12 it induces a smooth homomorphism

$$\mathcal{R}_{F_1/F'}\mathbb{G}_{m,F_1}^n \cong \mathcal{R}_{F/F'}\mathcal{R}_{F_1/F}\mathbb{G}_{m,F_1}^n \longrightarrow \mathcal{R}_{F/F'}T.$$

In particular, this morphism is dominant. On the other hand we have an open embedding  $\mathbb{G}_{m,F_1}^n \hookrightarrow \mathbb{A}_{F_1}^n$  and hence, by [4] A.2.11, an open embedding  $\mathcal{R}_{F_1/F'}\mathbb{G}_{m,F_1}^n \hookrightarrow \mathcal{R}_{F_1/F'}\mathbb{A}_{F_1}^n$ . It is trivial to show that  $\mathcal{R}_{F_1/F'}\mathbb{A}_{F_1}^n \cong \mathbb{A}_{F'}^{nd}$ , where  $d = [F_1/F']$ . It follows that the  $F'$ -rational points in  $\mathcal{R}_{F_1/F'}\mathbb{G}_{m,F_1}^n$  are Zariski dense, and so their images form a Zariski dense set of  $F'$ -rational points in  $\mathcal{R}_{F/F'}T$ , proving the theorem in this case.

If  $G$  is arbitrary let  $T$  be a maximal torus of  $G$ . As  $\mathcal{R}_{F/F'}T$  is commutative, it possesses a unique maximal torus  $T'$ , which is smooth over  $F'$  by [11] Thm.13.3.6.

**Lemma 1.8.**  $\mathcal{R}_{F/F'}T$  is the centralizer of  $T'$  in  $G'$ .

*Proof.* If  $F/F'$  is separable, this follows from the fact that  $\mathcal{R}_{F/F'}T$  is a maximal torus of  $G'$ . So assume that  $F/F'$  is inseparable of characteristic  $p$ . Since  $(\mathcal{R}_{F/F'}T)/T'$  is unipotent, we have  $T' = (\mathcal{R}_{F/F'}T)^{p^n}$  for suitable  $n \gg 0$ . As  $T'$  is smooth and the rational points of  $\mathcal{R}_{F/F'}T$  are Zariski dense, the centralizer of  $T'$  is equal to the centralizer of  $(\mathcal{R}_{F/F'}T)(F')^{p^n}$ . Note that the universal property of the Weil restriction identifies  $(\mathcal{R}_{F/F'}T)(F')$  with  $T(F)$ .

Consider a scheme  $S'$  over  $F'$  and an  $S'$ -valued point  $\varphi' : S' \rightarrow G'$ . Via the universal property of the Weil restriction  $\varphi'$  corresponds to an  $S' \times_{F'} F$ -valued point  $\varphi : S' \times_{F'} F \rightarrow G$ . We have seen that  $\varphi'$  factors through the centralizer of  $T'$  if and only if it commutes with  $(\mathcal{R}_{F/F'}T)(F')^{p^n}$ . This is equivalent to saying that  $\varphi$  commutes with  $T(F)^{p^n}$ . As  $T$  is a torus and  $F$  infinite, the subgroup  $T(F)^{p^n}$  is Zariski dense in  $T$ . The condition therefore amounts to saying that  $\varphi$  factors through the centralizer of  $T$ . But this centralizer is equal to  $T$ . Therefore, translated back to  $G'$ , the condition says that  $\varphi'$  factors through  $\mathcal{R}_{F/F'}T$ . This proves the lemma.  $\square$

By Lemma 1.8 the subgroup  $\mathcal{R}_{F/F'}T$  is the centralizer of a maximal torus of  $G'$ , i.e., it is a Cartan subgroup of  $G'$ . Thus [11] Cor.13.3.12 implies that  $G'(F')$  is Zariski dense in  $G'$ , proving Theorem 1.7.  $\square$

*Remark 1.9.* If  $F'$  is a non-discrete complete normed field, Theorem 1.7 is true for arbitrary connected smooth algebraic groups  $G$ . This is an easy consequence of the implicit function theorem.

Next we turn to simple groups. To fix ideas, a smooth linear algebraic group over a field will be called *simple* if it is non-trivial and possesses no non-trivial proper connected smooth normal algebraic subgroup. It is called *absolutely simple* if it remains simple over the algebraic closure of the base field.

If  $G$  is simply connected semisimple and simple over  $F$ , it is isomorphic to  $\mathcal{R}_{F_1/F}G_1$  for an absolutely simple simply connected semisimple group  $G_1$  over some finite separable extension  $F_1/F$  (cf. [11] Ex.16.2.9). From [4] A.2.4 we then deduce that  $G' \cong \mathcal{R}_{F_1/F'}G_1$ . In this way questions about  $G'$  can be reduced to the case that  $G$  is absolutely simple.

**Theorem 1.10.** Assume that  $G$  is simply connected semisimple and simple over  $F$ . Then  $G'$  is simple over  $F'$ .

*Proof.* By the above remarks we may assume that  $G$  is absolutely simple. Consider a non-trivial connected smooth normal algebraic subgroup  $H' \subset G'$ . Let

$$\bar{H}' \subset \prod_{\sigma \in \Sigma} G \times_{F, \sigma} E' \quad (1.11)$$

denote the image of  $H' \times_{F'} E'$  under the composite of the natural maps

$$G' \times_{F'} E' \xrightarrow{1,1} \prod_{\sigma \in \Sigma} G_{\sigma} \twoheadrightarrow \prod_{\sigma \in \Sigma} G_{\sigma}/G_{\sigma,1} \xrightarrow{1,2} \prod_{\sigma \in \Sigma} G \times_{F, \sigma} E'.$$

Since  $H'$  is non-trivial and “defined over  $F'$ ”, by [11] Cor.12.4.3 we have  $\bar{H}' \neq 1$ . Since  $H' \subset G'$  is a connected normal subgroup, so is  $\bar{H}'$  in 1.11. It is therefore equal to the product of some of the factors on the right hand side. As  $\bar{H}'$  is non-trivial, it contains at least one of these factors. But by construction it is also invariant under  $\text{Aut}(E'/F')$ , which permutes the factors transitively. We deduce that the inclusion 1.11 is in fact an equality. Now the following lemma implies that  $H' \times_{F'} E' = G' \times_{F'} E'$ ; and hence  $H' = G'$ , as desired.  $\square$

**Lemma 1.12.** *In the situation of Theorem 1.10, every normal algebraic subgroup  $H \subset G' \times_{F'} E'$  which surjects to  $\prod_{\sigma \in \Sigma} G \times_{F, \sigma} E'$  is equal to  $G' \times_{F'} E'$ .*

*Proof.* Using descending induction on  $i$  we will prove that  $G_{\sigma, i} \subset H$  for all  $\sigma \in \Sigma$  and  $1 \leq i \leq q$ . For  $i = q$  the assertion is obvious, because  $G_{\sigma, q} = 1$ . Let us assume the inclusion for  $G_{\sigma, i+1}$  and abbreviate

$$\text{gr}_i H_\sigma := \frac{H \cap G_{\sigma, i}}{G_{\sigma, i+1}} \subset \frac{G_{\sigma, i}}{G_{\sigma, i+1}} \stackrel{1.3}{\cong} \text{Lie } G \otimes_{F, \sigma} \mathbb{G}_{a, E'}. \quad (1.13)$$

By functoriality of the isomorphism 1.3, the conjugation action of  $G'(E')$  on  $G_{\sigma, i}$  corresponds to the adjoint representation of  $G \times_{F, \sigma} E'$  on the right hand side. As  $H$  is a normal subgroup, all commutators between  $H$  and  $G_{\sigma, i}$  must lie in  $H$ . It follows that

$$(\text{Ad}_h - \text{id})(\text{Lie } G) \otimes_{F, \sigma} \mathbb{G}_{a, E'} \subset \text{gr}_i H_\sigma \quad (1.14)$$

for every  $h \in H(E')$ . Since  $G$  is simply connected, it is known that the space of coinvariants of its adjoint representation is trivial (cf. [1], [2], or [5] Prop.1.11). On the other hand  $E'$  is algebraically closed, so by assumption  $H(E')$  maps to a Zariski dense subgroup of  $G \times_{F, \sigma} E'$ . Thus, as  $h$  varies, the subgroups in 1.14 generate  $\text{Lie } G \otimes_{F, \sigma} \mathbb{G}_{a, E'}$ . The inclusion in 1.13 is therefore an equality, and so we have  $G_{\sigma, i} \subset H$ .

At the end of the induction we have  $G_{\sigma, 1} \subset H$  for all  $\sigma \in \Sigma$ . Combining this with the fact that  $H$  surjects to  $\prod_{\sigma \in \Sigma} G_\sigma / G_{\sigma, 1}$ , we finally deduce  $H = G' \times_{F'} E'$ , as desired. This proves Lemma 1.12 and thereby finishes the proof of Theorem 1.10.  $\square$

*Remark 1.15.* The analogue of Theorem 1.10 fails if  $G$  is not simply connected and both  $F/F'$  and the universal central extension  $\pi : \tilde{G} \rightarrow G$  are inseparable. The reason is that by Proposition 1.6 (b) the homomorphism  $\mathcal{R}_{F/F'} \varphi : \mathcal{R}_{F/F'} \tilde{G} \rightarrow G'$  is not surjective, so its image is a subgroup that makes  $G'$  not simple.

**Corollary 1.16.** *If  $G$  is semisimple and simply connected, then  $G'$  is perfect.*

*Proof.* We may assume that  $G$  is simple. Then  $G$  is connected and non-commutative; hence so is  $G'$ . The commutator group of  $G'$  is therefore non-trivial connected and normal, and by [11] Cor.2.2.8 it is “defined over  $F$ ” and thus smooth. By Theorem 1.10 it is therefore equal to  $G'$ , as desired.  $\square$

**Theorem 1.17.** *If  $G$  is simple isotropic and simply connected and  $F$  is infinite, then  $G'$  is generated by split tori.*

*Proof.* By assumption there exists a closed embedding  $\mathbb{G}_{m,F'} \times_{F'} F \cong \mathbb{G}_{m,F} \hookrightarrow G$ . The homomorphism  $\mathbb{G}_{m,F'} \rightarrow G'$  corresponding to it by the universal property of the Weil restriction is again non-trivial; hence  $G'$  contains a non-trivial split torus. The algebraic subgroup of  $G'$  that is generated by all split tori in  $G'$  is therefore non-trivial. By construction it is normalized by  $G'(F')$ , so by Theorem 1.7 it is normal in  $G'$ . Being generated by smooth connected subgroups, it is itself smooth and connected by [11] Prop.2.2.6 (iii). By Theorem 1.10 it is therefore equal to  $G'$ , as desired.  $\square$

## 2. Main results

In the following we consider a connected semisimple group  $G$  over a local field  $F$ . Let  $\pi : \tilde{G} \rightarrow G$  denote its universal central extension. The commutator pairing  $\tilde{G} \times \tilde{G} \rightarrow \tilde{G}$  factors through a unique morphism

$$[\cdot, \cdot]^\sim : G \times G \rightarrow \tilde{G}.$$

For any closed subgroup  $\Gamma \subset G(F)$  we let  $\tilde{\Gamma}'$  denote the closure of the subgroup of  $\tilde{G}(F)$  that is generated by the set of generalized commutators  $[\Gamma, \Gamma]^\sim$ .

**Theorem 2.1.** *Let  $F$  be a local field, and let  $G$  be an isotropic connected simple semisimple group over  $F$ . Let  $\Gamma \subset G(F)$  be a non-discrete closed subgroup whose covolume for any invariant measure is finite. Then  $\tilde{\Gamma}'$  is open in  $\tilde{G}(F)$ .*

Before proving this, we note the following consequence (cf. [8], [3]).

**Corollary 2.2.** *Under the assumptions of Theorem 2.1 we have  $\tilde{\Gamma}' = \tilde{G}(F)$ . In particular,  $\Gamma$  contains  $\pi(\tilde{G}(F))$ .*

*Proof.* Since  $G(F)$  is not compact and  $\Gamma$  is a subgroup of finite covolume, this subgroup is not compact. Thus  $\tilde{\Gamma}'$  is normalized by an unbounded subgroup of  $G(F)$ , and it is open in  $\tilde{G}(F)$  by Theorem 2.1. As in [6] Thm.2.2 one deduces from this that  $\tilde{\Gamma}'$  is unbounded. Let  $\tilde{G}(F)^+$  denote the subgroup of  $\tilde{G}(F)$  that is generated by the rational points of the unipotent radicals of all rational parabolic subgroups. The Kneser-Tits conjecture, which is proved in this case (see [7] Thm. 7.6 or [10]), asserts that  $\tilde{G}(F)^+ = \tilde{G}(F)$ . On the other hand, a theorem of Tits [9] states that every unbounded open subgroup of  $\tilde{G}(F)^+$  is equal to  $\tilde{G}(F)^+$ . Altogether this implies  $\tilde{\Gamma}' = \tilde{G}(F)$ , as desired.  $\square$

*Proof of Theorem 2.1.* In the case  $\text{char}(F) = 0$  the proof in [8] §2 cannot be improved. It covers in particular the archimedean case. We will give a unified proof in the non-archimedean case, beginning with a few reductions.

Let  $\Gamma^{\text{ad}}$  denote the image of  $\Gamma$  in the adjoint group  $G^{\text{ad}}$  of  $G$ . Then  $\tilde{\Gamma}'$  depends only on  $\Gamma^{\text{ad}}$ . On the other hand, all the assumptions in 2.1 are still satisfied for  $\Gamma^{\text{ad}} \subset G^{\text{ad}}(F)$ . Namely, since the homomorphism  $G(F) \rightarrow G^{\text{ad}}(F)$  is proper with finite kernel, the subgroup  $\Gamma^{\text{ad}}$  is still non-discrete and closed. On the other hand, as the image of  $G(F)$  in  $G^{\text{ad}}(F)$  is cocompact, the covolume of  $\Gamma^{\text{ad}}$  in  $G^{\text{ad}}(F)$

is again finite. To prove the theorem, we may therefore replace  $G$  by  $G^{\text{ad}}$  and  $\Gamma$  by  $\Gamma^{\text{ad}}$ . In other words, we may assume that  $G$  is adjoint.

Next, since  $G$  is connected simple and adjoint, it is isomorphic to  $\mathcal{R}_{F_1/F} G_1$  for some absolutely simple connected adjoint group  $G_1$  over a finite separable extension  $F_1/F$ . If  $\tilde{G}_1$  denotes the universal covering of  $G_1$ , we then have  $\tilde{G} \cong \mathcal{R}_{F_1/F} \tilde{G}_1$ . By the definition of Weil restriction we have  $G(F) \cong G_1(F_1)$  and  $\tilde{G}(F) \cong \tilde{G}_1(F_1)$ ; and since  $G$  is isotropic, so is  $G_1$ . Thus after replacing  $F$  by  $F_1$  and  $G$  by  $G_1$  we may assume that  $G$  is absolutely simple.

For the next preparations note that  $F$  is non-archimedean, so  $G(F)$  possesses an open compact subgroup. Its intersection with  $\Gamma$  is an open compact subgroup of  $\Gamma$ ; let us call it  $\Delta$ . Let  $\tilde{\Delta}'$  denote the closure of the subgroup of  $\tilde{G}(F)$  that is generated by the set of generalized commutators  $[\Delta, \Delta]^\sim$ .

We will study the relation between these subgroups and various Weil restrictions of  $G$ . Consider any closed subfield  $F' \subset F$  such that  $[F/F']$  is finite. Note that in the case  $\text{char}(F) = 0$  there is a unique smallest such  $F'$ , namely the closure of  $\mathbb{Q}$ . But in positive characteristic the extension  $F/F'$  may be arbitrarily large and, what is worse, it may be inseparable.

Set  $G' := \mathcal{R}_{F/F'} G$  and  $\tilde{G}' := \mathcal{R}_{F/F'} \tilde{G}$ , and let  $\pi' : \tilde{G}' \rightarrow G'$  be the homomorphism induced by  $\pi$ . From Proposition 1.6 we know that  $\pi'$  is not necessarily an isogeny. Identifying  $G(F)$  with  $G'(F')$  via the universal property of the Weil restriction, we can view  $\Gamma$  as a non-discrete closed subgroup of finite covolume of  $G'(F')$ . Similarly, we can view  $\tilde{\Delta}'$  as a subgroup of  $\tilde{G}'(F')$ .

**Lemma 2.3.**  $\tilde{\Delta}'$  is Zariski dense in  $\tilde{G}'$ .

*Proof.* Let  $H' \subset G'$  and  $\tilde{H}' \subset \tilde{G}'$  be the Zariski closures of  $\Delta$  and  $\tilde{\Delta}'$ , respectively. By [11] Lemma 11.2.4 (ii) these groups are “defined over  $F'$ ”, i.e., smooth over  $F'$ . The intersection of  $\Delta$  with the identity component of  $H'$  is open in  $\Delta$  and thus again an open compact subgroup of  $\Gamma$ . After shrinking  $\Delta$  we may therefore assume that  $H'$  is connected. For any  $\gamma \in \Gamma$  the subgroup  $\gamma \Delta \gamma^{-1}$  is again an open compact subgroup of  $\Gamma$ , so it is commensurable with  $\Delta$ . Thus  $\gamma H' \gamma^{-1}$  is commensurable with  $H'$ . Since  $H'$  is connected, they must be equal; hence  $H'$  is normalized by  $\Gamma$ . It is therefore also normalized by the Zariski closure of  $\Gamma$ .

Under the assumptions of 2.1, a theorem of Wang [12] implies that the Zariski closure of  $\Gamma$  in  $G'$  contains all split tori of  $G'$ . Thus, in particular, it contains the images under  $\pi'$  of all split tori in  $\tilde{G}'$ . Since  $G$  is simple isotropic, so is  $\tilde{G}$ ; hence by Theorem 1.17 these tori generate  $\tilde{G}'$ . It follows that  $H'$  is normalized by the image of  $\tilde{G}'$ . By construction  $\tilde{H}'$  is the algebraic subgroup of  $\tilde{G}'$  that is generated by the image of the connected variety  $H' \times_{F'} H'$  under  $[\cdot, \cdot]^\sim$ . It is therefore connected and normalized by  $\tilde{G}'$ .

Since  $\Gamma$  is non-discrete, the group  $\Delta$  is not finite, and so  $H'$  is non-trivial. Let  $H$  denote the image of  $H' \times_{F'} F$  under the canonical adjunction morphism  $G' \times_{F'} F \rightarrow G$ . By construction  $H$  is just the Zariski closure of  $\Delta$  in  $G$ , so by the above arguments in the case  $F' = F$  it is normalized by the image of  $\tilde{G}$ . But  $\pi : \tilde{G} \rightarrow G$  is surjective, so  $H$  is a non-trivial connected normal subgroup of  $G$ . As  $G$  is absolutely simple, this implies  $H = G$ . As  $G$  is perfect, it follows that  $\tilde{H}' \times_{F'} F$  surjects to  $G$ .

All in all we now deduce that  $\tilde{H}'$  is a non-trivial connected smooth normal algebraic subgroup of  $\tilde{G}'$ . By Theorem 1.10 this implies  $\tilde{H}' = \tilde{G}'$ , as desired.  $\square$

Note that Lemma 2.3 in the case  $F' = F$  says that  $\tilde{\Delta}'$  is Zariski dense in  $\tilde{G}$ . In particular  $\Delta$  is compact and Zariski dense in  $G$ , so we can apply [5] Main Theorem 0.2. It follows that there exists a closed subfield  $E \subset F$  such that  $[F/E]$  is finite,  $E$  is absolutely simple and simply connected semisimple algebraic group  $\tilde{H}$  over  $E$ , and an isogeny  $\tilde{\varphi}: \tilde{H} \times_E F \rightarrow \tilde{G}$  with non-vanishing derivative, such that  $\tilde{\Delta}'$  is the image under  $\tilde{\varphi}$  of an open subgroup of  $\tilde{H}(E)$ .

**Lemma 2.4.**  $E = F$ .

*Proof.* Via the universal property of the Weil restriction the isogeny  $\tilde{\varphi}$  corresponds to a homomorphism  $\tilde{\varphi}': \tilde{H} \rightarrow \mathcal{R}_{F/E}\tilde{G}$ , which satisfies

$$\tilde{\Delta}' \subset \tilde{\varphi}'(\tilde{H}(E)) \subset (\mathcal{R}_{F/E}\tilde{G})(E) = \tilde{G}(F).$$

By Lemma 2.3 in the case  $F' = E$  we know that  $\tilde{\Delta}'$  is Zariski dense in  $\mathcal{R}_{F/E}\tilde{G}$ . It follows that  $\tilde{\varphi}'$  is dominant. This implies

$$\dim \tilde{H} \geq \dim \mathcal{R}_{F/E}\tilde{G} = [F/E] \cdot \dim \tilde{G} = [F/E] \cdot \dim \tilde{H};$$

hence  $[F/E] = 1$ , as desired.  $\square$

**Lemma 2.5.**  $\tilde{\varphi}$  is an isomorphism.

*Proof.* As  $\tilde{\varphi}$  is an isogeny between simply connected groups, it is an isomorphism if and only if it is separable. In characteristic zero this is automatically the case. (Since  $d\tilde{\varphi} \neq 0$ , this is actually true whenever  $\text{char}(F) \neq 2, 3$  (cf. [5] Thm.1.7), but we do not need that fact.) So for the rest of the proof we may suppose that  $p := \text{char}(F)$  is positive. Set  $F' := \{x^p \mid x \in F\}$ ; then  $F/F'$  is an inseparable extension of degree  $p$ . Consider the induced homomorphism

$$\tilde{\psi} := \mathcal{R}_{F/F'}\tilde{\varphi}: \mathcal{R}_{F/F'}\tilde{H} \longrightarrow \mathcal{R}_{F/F'}\tilde{G}.$$

By construction it satisfies

$$\begin{array}{ccc} \tilde{\Delta}' & \subset & \tilde{\psi}((\mathcal{R}_{F/F'}\tilde{H})(F')) \subset (\mathcal{R}_{F/F'}\tilde{G})(F') \\ & \parallel & \parallel \\ & \tilde{\varphi}(\tilde{H}(F)) & \subset \tilde{G}(F). \end{array}$$

Since  $\tilde{\Delta}'$  is Zariski dense in  $\mathcal{R}_{F/F'}\tilde{G}$  by Lemma 2.3, we deduce that  $\tilde{\psi}$  is dominant. So for dimension reasons it is an isogeny. Proposition 1.6 (b) now shows that  $\tilde{\varphi}$  is separable, as desired.  $\square$

Combining Lemmas 2.4 and 2.5, we now deduce that  $\tilde{\Delta}'$  is open in  $\tilde{G}(F)$ . Thus  $\tilde{\Gamma}'$  is open in  $\tilde{G}(F)$ , completing the proof of Theorem 2.1.  $\square$

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